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Time and Black–Scholes–Merton

No statistics can represent the abyss of the event. Only the market can, provided the story is told right.

The formalism of Black–Scholes–Merton knows of no such thing as the past or the future. When it models the stochastic process of the underlying asset price as Brownian motion and symbolizes its volatility by σ, this is just 'the volatility,' a formal concept, and *t* is formal time. Physical time and physical history are just an interpretation of the timeless formalism. Likewise, statistics is an interpretation of the probability formalism. The formalism of Kolmogorov does not recognize the existence of a material dice that one is casting over and over again. The formal proof of the strong law of large numbers recognizes a series of formal random variables, each corresponding to a different throw of the dice, and it recognizes that the throws are independent and have the same probability distribution. However, it is a step outside the formalism to reunite those throws under the same matter and to assume the existence of a single dice that one is throwing, collecting, lifting, and throwing again, chronologically. Matter as such has no existence in the formalism. It is we who interpret the sequence of random variables as *trials*; that is to say, as repeated experiments – as trials of the same invariable matter.

Of course, nothing stops us from considering the time series of prices of the underlying asset in comparable fashion – for this series undoubtedly exists. However, when we realize that no such historical series can have any purchase on the value of the forward-looking derivative... better, when we realize that the whole register of time, in which such a time series is conceived (regardless of whether we conceive it as extending in the past or in the future), is not the same register of time as the one in which the market price of the derivative is given,[1] we realize something both difficult and deep, which is that the nature of discourse pertaining to statistics and to time series is different from, and is even incompatible with, the nature of discourse pertaining to the derivatives market. This is despite the fact that the actual time series of underlying prices and the actual market prices of derivatives written on that underlying seem to occur in the same physical reality and in the same physical time. As a matter of fact, the difficult and counterintuitive distinctions that we are trying to make, in thus speaking of different 'registers of time' or different 'natures of discourse', can be simply enacted by ruling that the market price of the derivative is a different interpretation of the formalism of BSM and of volatility – that is to say, a step outside the formalism in a different direction – than the time series of underlying prices.

Volatility as the meaning of price

Volatility is a statistical concept and, in practice, you need an extended time series of prices of the underlying asset in order to estimate it; yet, the advent of BSM has made it so that the volatility of the underlying asset price is inferable from the single traded price of the derivative written on it, by inversion of the BSM formula. Instead of asking a statistician or an econometrician to hand us the volatility of the underlying asset price in order to use it as an input in the BSM formula and get the value of the option as an output, we read the option price from the options market, and we invert the BSM formula against it, in order to imply (or reverse-engineer) the value the volatility must have, which we subsequently call *implied volatility*. We can now estimate a whole statistics i.e., volatility – from a single datum.

Everybody thinks this is just an artifact of the formalism of BSM and of Brownian motion, and that the real volatility of the underlying asset price still has to be inferred statistically, in reality. However, in view of the distinction we have made above, between the register of time in which statistics are given and the register of time in which the derivative market price is given, it no longer appears to us as an artifact of the formalism that the volatility should be implied from the single option price instead of the extended time series of the underlying price; and it is no longer true that volatility still has to be inferred statistically, in reality. Rather, the reality of the option market price now appears as an *alternative reality* to statistics. Instead of stepping out of the formalism in the usual direction and encountering the usual notion of matter – the matter of the dice, or the material random generator which we unconsciously extrapolate beyond limits and proportions when we believe that there exists such a one in the market, lying behind the underlying prices – instead of stepping out in that direction, we step out in the opposite direction, in which the derivative price is the only 'random generator' of the underlying price, and in which the

latter no longer follows a stochastic or temporal process but follows, instead, its writing as a derivative, whose price follows, in turn, its writing as a derivative of next level, etc.

A price series, indeed, of derivatives of increasing complexity; all immersed in the market and all composing a single compact matter, instead of a time series of prices of a single underlying. A very difficult thought indeed; for it is difficult to realize that time is not the only choice, when it comes to perspective. Doubtless, this shift of perspective, which is truly metaphysical – and by that, I mean that it won't change the physical world or physical reality, since all that one can observe, in physical time and physical reality, will still be the single register in which both the underlying asset and the derivative instrument trade empirically and register their time series of prices, a register in which the majority of quantitative analysts are still trapped – doubtless, this shift of perspective will have consequences, on the notions of prediction and on the ways of dealing with the contingent future and with the event, which we will have to draw in full in order to show the extraordinary matter or fabric that the market is made of.

Notice that we are not only shifting the perspective and shifting the reality of the market from statistics to derivatives trading, or from time to place (the marketplace), but that we are, above all, changing the meaning of reality. What reality of the market could we be talking about – one may ask, indeed – if not the one and only physical reality? What reality could be that one, in which time may admit of different registers and such that, if you embarked in one, you could no longer mix it with the other, on pains of inconsistency? Inconsistency of what? If physical reality (the one and only reality) is one in which trading happens, regardless of whether it is the trading of the underlying or of the derivative, then how could a physical theory of the market not address, and only address, the workings of such trading? And, in case it finds that the physical causes of trading are too numerous and multifarious to be modeled mechanistically (that there is an infinity of causes, as Bachelier (1995) would say), how could it not resort to statistical analysis? How could the physical theory of the market not ultimately come down to proposing stochastic processes, or random generators, for any price, regardless of whether it is the price of an underlying or a derivative? And how could that be inconsistent? Isn't physical reality supposed to be consistent, no matter what?

I don't believe it is the empirical nature or the empirical existence of the market – what the market is, empirically – that really matters here but, rather, the way we think of it. The market should be approached in the same way as we approach a book, or an entity in which thought claims a constitutive part. This is a new kind of genesis, a new kind of reality altogether, which entertains an unusual link with theory and fiction and necessitates a whole new kind of philosophical analysis. I am not saying that the market is a fabrication of man and, as a consequence, should only be left to anthropologists or to sociologists to analyze its matter and spell out its ontology. Rather, the market deals with the event and with the contingent future, which are real but are perspectival and engage man in their *making* rather than in their detached or 'objective' observation. Neither am I suggesting that the market is a matter of psychology or subjective belief. It is of no help to say that the market is an aggregate of subjective beliefs, for that vision of the market also occurs from outside and uses an ill-defined notion, the belief, to compose an even more ill-defined notion, the aggregate of beliefs. The market is a trading arena, in which anything can happen and in which the trader faces, at any instant, the abyss of the contingent event.

It is true that the trader in the pit seems to face, at any time, no greater an event than the up or down movement of the trading price and the magnitude of that movement, and it is true that, thanks to the identifiable existence of that seemingly elementary event, the tendency has been to look at the market exclusively through the quantitative lenses of statistics or probability. In reality, what this next movement of price – what the next tick of price and of any price – contains is the whole event and the whole abyss, because it contains, as we shall see, the virtual exchange and the virtual pricing of *all the derivatives that are virtually written on the underlying asset, and written successively, one upon the other*.

A price, even the price of the most basic asset, has no meaning separately from its volatility. The price of an asset, trading freely in an open and frictionless exchange, is a price only insomuch as the traders exchange the asset in anticipation of a future rise or fall of its price, and only insomuch as those future movements are in themselves totally unpredictable. For, if a future price were predictable, the asset wouldn't trade at that price in the future but on the spot, by the sheer competition between traders, leaving for the future only the unpredictable. So, volatility becomes the only certain thing and the only conclusion; volatility becomes the only fundamental value; volatility becomes the commodity and the true stuff that the market is made of. Suddenly, science and quantitative theory move from trying to determine the value of the traded asset, which is impossible because the only thing that exists is its market price, to determining (that is to say, to imagining, to picturing, to conceiving) the volatility of its price. Let volatility be σ.

Volatility is not a price; it doesn't belong to the same semantic realm or plane of reference as the price

Volatility is not a price; it doesn't belong to the same semantic realm or plane of reference as the price. God sees and knows volatility. So, volatility is known. Volatility is. When we talk of the market, when we exert our philosophical gaze upon it, when we *conceive* of the market, we conceive of the volatility of the market price. Volatility belongs to the concept of the market. Even if it is inaccessible, empirically, volatility is very clear and even certain, semantically. The reasoning above, to the effect that price had to be random in order to be at all, wasn't quantitative and didn't introduce a particular quantity, known as volatility, or a particular quantitative stochastic process. It showed randomness to belong to the same level as the *meaning* of price. Therefore, when we model the behavior of the price as Brownian motion, obviously, the conceptual

level to which we rise and in which we think of that stochastic process and of its volatility is not the same level as the price whose motion we are modeling. We are not modeling the empirical reality of price, or providing a statistical summary of its motion; we are modeling its meaning.

BSM is the first absolute case of valuation of something (the derivative) because the market (of the underlying asset) is the case where valuation is absolutely out of the question and only the price prevails, and because *this* observation is the absolute appraisal or *evaluation* of the market. Price is the ontological opposite of value, and it is important to realize that BSM has created and determined value (of the derivative) *through and out of* the very constitution and meaning of price (of the underlying), which is that price is essentially volatile and essentially contrary to value, and not on the side of price or in competition with price. So, we should really speak of the 'valuation' of derivatives in BSM as the *evaluation* of the underlying market or a conceptualization of the volatility of the underlying price, not as valuation in the ordinary sense. It is important to realize this philosophical revolution of BSM and the intertwinement of the levels of thought that it implies. It is important to understand the feeling of power and certainty (if unconscious) with which traders came back to the options pits, armed with BSM.

Meta-theoreticians of BSM (either sociologists of finance or critics writing from within the field of derivative pricing theory) are still confused about the origin and being of options markets. They believe that options trade on the floor because volatility is uncertain and is not known by traders. They think the volatility is the new thing that we have no choice, either, but to leave to the market to determine; that it is the new price, and that the option value, which is a function of volatility, subsequently becomes a price, trading on the side of the underlying price. It is true that one needs to know volatility in order to use the BSM formula in practice. However, the ascent that I suggested was taking place between levels of thought, or the fact that BSM was a case of absolute valuation – what I called an *evaluation* – precisely because it didn't compare or compete with price but rose above it, because it rose to the concept of price and entered its very constitution and meaning as volatile precisely, this ascent is independent of knowledge or epistemology. It is semantic. It has to do with the meaning and constitution of things.

We look at the binomial tree from outside and we mock its simplicity, but to the creatures from inside that world, to traders still immersed in that pit, truly, anything can still happen, even when modeled

To repeat, when one rises above the floor of the underlying market and formally writes the process it is following – in this case, Brownian motion with volatility σ – this process and this volatility become given; they become symbols that have no particular numerical values, because they are mathematical symbols. That they should have no numerical value does not mean that we don't *know* the numerical value. Knowledge has nothing to do with this. Of course, the volatility we are using in BSM will have to be equal to something or other; however, this 'something or other' is not a variation that is due to uncertainty or to randomness. Surely, σ will have to be equal to some number; but what it is, what its value is, is σ. This is a constant, because the symbol is constant. Even if the volatility σ were a function of time t, $\sigma(t)$ would remain a constant symbol.

Volcanic time vs. statistical time

For anyone immersed in the trading pit of the underlying asset, time is pure event and is certainly not open to statistical accountancy. Time is pure creation time, and anything can happen in the next instant. The trading pit lives in history, and history is full of events. Any such historical event has an impact on the trading pit; however, its only visible translation, in the trading pit, is the price going up or going down. When our philosophical gaze steps outside the pit to represent the time of the pit, to say what the pit is, and due to the seemingly quantitative aspect of its events (namely, that the price is seemingly a number and can only go up or down), we have no choice but to quantitatively *model* the behavior of the price – for instance, as Brownian motion.

It is important to realize that this is a model. It is a model of the pure creation time and of the volcanic time in which anything can happen. It is a model and, as such, it is simple. However, as quantitatively simple as it may be, it is not supposed, as a model of time, to change the *nature* of time. We are confused by the nature of number and with the way we name the quantities. It is not because the binomial tree presents the price only with two states of the world, or because its limit in continuous time, Brownian motion, presents the price only with a diffusion, that the time of trading – the time from inside the pit, which we have called volcanic time – will suddenly vanish and be taken over by the simplistic mechanical clock of the random walk, in which, apart from flipping a coin and oscillating between two states, nothing new can happen anymore.

To repeat, we may have no other means of representing volcanic time and innovation, in our toy model, than by two states; but this limitation of the quantitative model should not change the *quality* of volcanic time. We look at the binomial tree from outside and we mock its simplicity, but to the creatures from inside that world, to traders still immersed in that pit, truly, anything can still happen, even when modeled. To their eyes, truly, the event is still an abyss (except that, in their world, there are only two states). It may seem that this distinction that I am trying to make between two perspectives and, correspondingly, between two registers of time – a distinction between internal volcanic time and the external clock of Brownian motion – is mere wordplay and makes no real difference; however, as we shall see shortly, the event of the derivative will make all the difference and recreate the abyss of the event in another place and dimension than the interval of time.

As seen through the eyes of traders from inside, the unquantifiable, the intractable, the unrepresentable, the unprecedented, or what we have called the *event*, is still happening at any moment, even when modeled as a binomial tree or as Brownian motion; they are still staring at it. Go ahead and imagine the multitude of things that could happen on the floor during a trading day; go ahead and remember the things that you saw happening, when you were a trader on the pit. Now, all of this is happening and is still happening – we are still stuck inside it– yet, the power of thought is such that we can step outside the pit to try to represent it or summarize it under a simple stochastic process, *while still hanging on the thought that it is the pit and what goes inside the pit*, namely the abyss, that we are thus representing.

Thought has the power to resist the illusion of its own representations. We summarize the trading activity (all that is happening and still happening) with a simple stochastic process – we have no other choice, when we exit into representation time – however, we must be careful, now, how we handle this summary in time. It is a summary; it is a conclusion; when we draw it, we pull it outside of time and time can no longer change it; however, the danger is that we might be misled by the quantitative character and quantitative variety of the stochastic processes. Because we can vary Brownian motion and make it more complex by assuming that its volatility can become stochastic or that jump processes can be superimposed on it, this conceptual concatenation misleads us into thinking that it is happening in time. We feel that we could re-enter the trading floor and re-enter its time, and the danger, now, is that we might look at price statistically, as if we were now waiting beside the pit, and *observing* whether volatility was going to change stochastically, or jumps were going to occur, and so on.

The danger is that we re-enter the trading floor, only we don't take part in its trading action anymore. We now stand beside the pit and we act as statisticians who merely record the time series of prices produced by the pit and who study their statistical properties. Suddenly, we observe that the volatility is no longer constant but has changed randomly; suddenly, we observe that jumps occur. We may very well do that, but, in reality, we would be confusing the registers of time. When we had first exited with the summary of the trading activity and represented it as a stochastic process, we, in thought, had gone all the way up *to the end of time*. Anything could happen and a multitude of unimaginable things could indeed emerge before the end of time, yet we had exited from that end and represented the summary by a stochastic process that nothing, no matter how unimaginable, could change any longer. The stochastic process occurs at the conceptual level, we said, the one we have called the *evaluation*, or the appraisal, of the market and of volatility. It is the conceptual summary of the trading pit and of its volcanic time.

Exiting from the pit should be forbidden, if thought doesn't have the power to separate the registers again. It should be forbidden to offer, as a summary, a stochastic process whose time would tempt us into thinking that it was now taking place beside the pit, registering the changes of price.

But then, the phrasing was incorrect, from the beginning! From the beginning, there has never been a derivative price!

When the trading activity is abandoned and the stochastic process is thought, mistakenly, to still occur in the same time, except that it is no longer volcanic time (as if the difference between volcanic time and statistical time was only a matter of stepping to the side of the pit), when thought fails to distinguish the registers and thinks that volcanic time and statistical time are two different descriptions of the one and the same physical time, and not two different natures of time, the temptation and hence the danger are great to re-enter the space of the market, no longer with trading or pricing purposes, but with the purpose of *valuation* – valuation in the ordinary sense. By this, we mean that we would no longer be attending to the *meaning* of price and of trading (what we have called the *evaluation* of the market, happening after the end of time) but would merely be observing its behavior and registering its price statistics. Now, the market becomes identified with a random generator and lotteries that pay out contingent amounts, depending on the underlying price, are imagined. Derivatives now become confused with those lotteries and we imagine that we can value them. Suddenly, the space of the market is re-entered as one in which probability replaces trading, statistical time replaces volcanic time, and the value of lotteries, up to and including the value of the underlying asset itself, replaces price.

The valuation perspective

It is commonly believed that BSM provides a dynamic trading strategy in the underlying asset which replicates perfectly the payoff of the derivative, with the consequence that the price of the derivative becomes equal to the cost of the replication strategy, and therefore is determined. Or is it the *value* of the derivative? How could the *price* of the derivative be determined? To mention a price presupposes that the derivative is delivered to the market, to its own market, and is subject to trading forces that pull it in directions, and for reasons, that are proper to it; so, how could the trading strategy in the underlying asset ever track that motion or curb it? It is believed that the derivative price doesn't have proper forces that pull it other than the motion of the underlying price; as a matter of fact, the derivative price is defined, from the start, as a function solely of the underlying price and time, in BSM. If volatility is constant, as in BSM, then there is no possible independent variability of the derivative price. But then, the phrasing was incorrect, from the beginning! From the beginning, there has never been a derivative price! It is not the case that the derivative price errs in its market and then the replication strategy pulls it back in line with the BSM prescription. Neither has the derivative market progressively adjusted itself to the BSM prescription, after generations of arbitrageurs, who were clever enough to use the BSM algorithm, had made money off the back of naïve derivatives traders (for we wonder how the clever ones were ever selected). There is no time for such a narrative in BSM. Such time only exists in sociological narratives. The BSM formula is semantically instantaneous, even though it stages

time, formally. It means one thing and it means it at once. It cannot mean both that the derivatives market exists and that it doesn't exist. So, from the start, there never was a derivatives market in BSM.

The only market there is, in BSM, is the market of the underlying asset. It is there that trading forces exist and that the abyss opens at every trade. However, the conceptual exit, which allowed us to represent that trading action once and for all, and to summarize it under an external stochastic process, allows us concomitantly, thanks to a dangerous equivocation between evaluation and valuation, to reapproach the pit and stand on its side, with a view to a probabilistic *valuation* of the derivatives. The tradable underlying asset turns into a random generator, into the mere trigger of mere lotteries, and the question turns into valuing those lotteries. As a matter of fact, the underlying asset itself is now confused with one such lottery, with the only difference that its value is known and is equal to its market price. When seen from outside the pit, it now appears merely as a coincidence that the value of the underlying asset should be equal to its market price. The principle of non-arbitrage applies, and it imposes on us that the value of all other lotteries triggered by the same device – namely, the derivatives – should be formally identical with a discounted expectation under an equivalent probability measure.

Definitely, valuation takes place outside the trading pit, beside it. It now only borrows from the trading pit the *value* of the underlying asset, which it knows is equal to the traded price, and it now respects the non-arbitrage space that it shares with the trading pit. However, it has definitely exited the latter's internal, volcanic time. Of course, the formalism doesn't care about such fine distinctions, and doesn't even see them. Of course, there is no other choice, formally, but to write an external stochastic process and to apply the principle of non-arbitrage. But is the point of view of the market, the view from inside the pit, only ever compatible with the outside point of view, in which the stochastic process is observed and an equivalent probability measure is found and selected in a completely detached manner? This is probably the most important question of the philosophy of quantitative finance.

It is believed that dynamic replication of the derivative's payoff under the BSM assumptions is what determines its value uniquely, but, in reality, if we reason from outside the pit, purely with equivalent martingale measures and from the detached valuation perspective, we realize that the pit's volcanic time has long been abandoned and that the dynamic trading activity in the underlying asset, which is supposed to produce the replication of the derivative, has lost its genuine force and is no less a loan taken from the pit than the value of the underlying asset, which is now only coincidentally equal to its price. Indeed, the volatility of Brownian motion is invariant by the Girsanov theorem of the change of measure, and the condition that the discounted expected value of the underlying asset must be equal to its market price is what picks the equivalent martingale measure uniquely and determines the derivative value uniquely. So, there never was a market in the valuation perspective, either of the underlying asset or of the derivative (we have long exited from the underlying pit), but only a series of coincidences. It is a coincidence that volatility should be preserved under the change of measure; it is a coincidence that the lottery whose outcomes are the future market prices of the underlying asset should admit as present value the present price of the underlying asset; and it is a coincidence that these two conditions should uniquely determine the equivalent martingale measure, when the motion is Brownian in the objective measure.

The register will be totally incompatible with the accounting register in which the time series of prices of the underlying asset is conceived

In the valuation perspective, trading is no longer a breath-taking activity, which could only be trusted to a living agent breathing inside the abyss of the pit, but is now analyzed as the coincidence between two detached parts – namely, that something, the price of the underlying, should at the same time be the lottery and the ticket to purchase it – in a unique configuration which we now *externally recognize* as being the circumstance of trading and of the market. That the underlying asset should be tradable is now only recognized as an incidental feature of the valuation problem, and is no longer the main original feature. There is no difference between such a situation and one in which a trading algorithm, standing beside the pit of the underlying asset, now executes buying or selling orders every time the holdings of the underlying asset need to be rebalanced. Dynamic trading becomes a mere expedient. Instead of wondering how to find the equivalent measure in the space of measures, the self-financing dynamic trading in the underlying asset now allows us to construct it, and the martingale representation theorem of Brownian motion finishes the proof of its unicity.

Such are the devastating effects of the external valuation perspective on the once un-exchangeable trading spot and of its algorithmic logic on the once volcanic time. On close scrutiny, however, it will soon appear that this perspective doesn't respect what BSM says exactly.

The letter of BSM

One way of preserving the nature of the conceptual exit, and of not confusing it with a statistical exit in which the pit is now observed from the side and becomes identified with a mere random generator, merely triggering lotteries and suggesting the mere valuation of lotteries, is to go along with what BSM is saying exactly, and to exploit just that.

We said that the conceptual exit from the trading pit did not want to compromise its volcanic time. Yes, a stochastic process for the sake of representation; yes, a stochastic process that formally had no choice but to feature time, yet a stochastic process that was meant to model volcanic time, to be the end of time, and to continue to harbor the abyss of the event and all that can happen. The trading action, in which anything can happen any-

time and in which the abysmal event always looms, entails randomness, of course, and this, when time is chosen as the framework, can only be *represented* as a stochastic process. But time is just one possible choice of representation. We have to keep in mind this arbitrariness of the choice of time, in order not to come back to the side of the pit, with the confusion that the stochastic process supposed to model the event and its volcanic time can now act as an accounting device in statistical time and help us value derivatives, now understood as mere lotteries written on the side of the pit.

The idea, here, is that when the time comes and the derivative is truly given in its market, not as a prewritten lottery but as a true tradable asset, *the register will be totally incompatible with the accounting register in which the time series of prices of the underlying asset is conceived*. The existence of a derivatives market, in which the event can be inscribed and written, is a different dimension than the time dimension. The time dimension lures us into predicting the event. When we leave to the event no other dimension to occur into than time and its actualization in time, there emerges the contradiction between the stochastic process, which seems to box and account for everything that will happen in time, and the event which can, then, only exceed that frame and spill outside the box. Time is not the right dimension. To really show how the mere imagination of a derivatives market is incompatible with statistical time and with the accounting or register of the time series of underlying prices, it suffices to go back to what BSM says exactly and to try to keep in mind the volcanic time, which, to repeat, it is only trying to model but not to deny.

Anyone who really trades inside the pit always trades in a given size and not only at a given price. Perhaps the best way of regaining the unexchangeable spot in which the living trader stands and makes actual decisions is to think of a dynamic trading strategy he would be following, of decisions he has continually to make on the spot, in the pure originality of the instant, right on the edge of the abyss. (Every option trader knows that the gamma vs. theta arbitrage is something that grabs him in the guts, and cannot be geometrized. McTaggart (1908) once expressed this, in the analysis of time, by distinguishing between A-series and B-series of time.) Trading is not just a stochastic process that one can register and write from outside. Trading mixes the variable size of the transaction with the random price, and one way of reassociating the subject with the event and of reintroducing, in the abyss of the event, the living flesh that the logic of the algorithm and of statistics seemed to have retired of late, is to summarize and model the volcanic time of the trading pit – what we have called the 'pure time of the event' – no longer with an abstract stochastic process but with a self-financing dynamic trading strategy, or, in other words, by a dynamic trader immersed inside that pit. This is exactly what BSM accomplishes.

When the letter of BSM is strictly followed through, in a rigorous paper, like Harrison and Pliska (1981) rather than Black and Scholes (1973), one finds that only the underlying asset and the pit in which it is trading are ever considered (together with the money account). No independently written derivative has ever been in sight, and even less so any intention of valuing it or replicating its payoff or trading it in its own market. All that BSM establishes is the dynamic trading strategy that the trader must follow in order to 'manufacture for himself' *contingent payoffs*, not contingent claims. BSM instructs the trader of the exact amount of money to invest, at the inception of the trading strategy, in the exact fraction of the underlying asset, then of the exact way of varying that fraction, as time goes by and the underlying asset price moves randomly, in order to end up, at the exit time or the maturity of the strategy, with the exact amount of money that the trader had in mind, at the start, as a function of the price that would then prevail for the underlying asset – in other words, a contingent payoff.

It is true that the fraction of the underlying asset that the dynamic trader needs to hold at any point in time and space – what is known at the derivative's delta – depends chiefly on his knowledge of the volatility of the underlying asset price; therefore, the question can be asked again of how such a dynamic and living trader, immersed in the trading action and in volcanic time, could ever have knowledge of a statistical parameter such as volatility. Aren't we confusing again the registers of time? Precisely, the answer lies in the way we have now set up the problem, after reading carefully what the formalism says. Earlier, we suggested that giving the derivative in its market, in the sense of immersion in trading and in volcanic time, is incompatible with the outside point of view in which the trading registers as a statistical time series of prices, and in which an algorithm is supposed to replicate the derivative's payoff. Now, we are saying something even more precise. We are saying that even when no derivative or derivatives market exists and we are only trying to manufacture a contingent payoff, or the payoff of a derivative or contingent claim *that has not yet been written independently*, even this activity is illicit, because it relies on the statistical register that is not available to the volcanic trader.

The BSM formalism is no less a conceptual exit from the trading pit of the underlying asset than the stochastic process that we described earlier. As a matter of fact, it is a more complete conceptualization of the trading activity, because it not only represents the randomness of the price, but also the variation of the size of the transaction. BSM never said what the value of the volatility was, or indicated how one would get it. Volatility is still a symbol in BSM – as much a symbol as when we said: Let the volatility be σ. And it is the continuation of the symbolism and of the formalism to go ahead and imagine the dynamic trading strategy whose actual unfolding depends on volatility, and consequently to attach a premium to a contingent payoff. This is our complete reading of the letter of BSM: the dynamic trader doesn't have access to statistical volatility, and we need not worry ourselves about that, because neither we nor the trader have exited in statistical time yet, or in representation for that matter. The trader only sees his trading; he only stares at the abyss; he cannot stand to measure or quantify volatility; he cannot engage in that register of time; we all agree on that. However, the conceptual exit from the abyss and the attempt to represent it is no longer achieved, as before, by Brownian motion and volatility alone, but by the completed story of both the price motion and the variable size of the trade. It is the premium to replicate the contingent payoff, or the completed BSM procedure, that will now stand as the conceptual exit.

Remember when we said that thought had the power not to yield to its representations and to consider that Brownian motion and volatility, with which it represented the 'absolute value' of trading, though quantitative as they may be and though temporal as they may seem, are absolute and really

belong to the end of time, to the horizon where anything that could happen has already happened (an absolute and an end of time that no relative variation can alter anymore, such as the idea that volatility may be different and that it may vary in time). (How could it differ, when it is a symbol?) Now, the algorithm of BSM is making things even easier; or, rather, the algorithm of BSM is how we should have, from the start, always looked at the conceptual summary of the trading activity of the underlying asset. Now, the conceptual summary or the exit from the trading activity is represented by a single premium attaching to a given payoff. Instead of imagining, as a conceptual summary of the volcanic activity, the whole Brownian process with its quantitative volatility and the corresponding expanse of time which precisely bears the risk of re-entry in time, now, we can summarize the trading activity with timeless or instantaneous quantities – with the initial premium that is needed to manufacture the contingent payoff and the contingent payoff itself, namely, the maturity T and the payoff function, for instance Max (0, $S - K$). And now, something extraordinary will happen, which, as we shall see, *is essentially due to the timelessness of the summary*.

The event of writing the contingent claim

Remember that the association of the premium with the contingent payoff is as absolute a conception and a summary of the trading activity of the underlying asset as the whole stochastic process. It is as certain and as absolute for an immersed trader that his market exists and should be volcanic as it is that volatility should be (and should be symbolized by σ) or that the premium to manufacture the contingent payoff should be (and should be symbolized by π). Both σ and π are conceptualizations and belong, as we said, to the realm beyond the end of time. Of course, in the projective imagination of that concept, σ has to be equal to something or other, and so does π. However, these variations, we said, are not really quantitative; they do not belong to the realm of physical time and numbers. Now, what happens next – a true event – is that the contingent payoff *gets written as a contingent claim*.

The trader who is immersed in a pit, who has every conceptual and semantic certainty of his volcanic trading activity, therefore every conceptual and semantic certainty of manufacturing the payoff for an initial premium π, translates this certainty as the certainty of making the market of the corresponding contingent claim and charging for it the price π. Two transfers are made – literally, two translations in space – as if an imprint was transferred from one material to the other. First, the contingent payoff materializes as a contingent claim and is written as one; the writing of the contingent claim is *invented* (for it amounts to the same to say that the contingent payoff is guaranteed to be manufactured no matter what, and to say that it is guaranteed by writing, as a contract binding a seller – that is, as a contingent claim). Second, the premium to manufacture it is translated as a price.

How the market they are making will become an autonomous creature is through the event we have described, or the unbridgeable difference between the contingent payoff and the contingent claim

Option market-makers, we said, did not step into their market-to-be with a sense of uncertainty or stochasticity of volatility. To the contrary, they stepped in with the certainty of their tool, even with an absolute value, what we have called the concept or the evaluation of their market. One cannot make a market – one cannot make anything – with uncertainty in mind. Options market-makers *know* volatility for certain and they *know* for certain how to manufacture the contingent payoff. What will vary and how the market they are making will become an autonomous creature is through the event we have described, or the unbridgeable difference between the contingent payoff and the contingent claim, *which is not a difference between constant volatility and stochastic volatility or between certain volatility and uncertain volatility*.

In a sense, it is much better to reimmerse the conceptual exit from the trading pit of the underlying into the trading pit of contingent claims written on the underlying – for, then, this would be going back inside the pit, completing, as we shall see, its record (memory) of the event, opening it to the event in its fullest – than to come back, only to stand beside the pit and produce what can only turn into statistical analysis of the underlying asset price or valuation of derivatives that are prewritten on it.

Indeed, now we can see how the market, which is swallowing everything back, will ultimately get rid of the dimension of time, which was only causing trouble to the representation of its volcanic activity and to the representation of the event. Earlier, we said that it was a conceptual and even a category mistake to come back to the side of the market, or to re-enter time, after the conceptual exit that was represented by Brownian motion. The conceptual exit was the absolute summary, and it could not be relativized and analogized, brought back in the world either to value lotteries (derivatives written on the side) or to monitor the statistics of the underlying price, eventually to receive criticism from them. (It is the view from inside, which cannot be exchanged with a view from beside, yet from which we can exit conceptually, as if from God's point of view.) But now, the situation is different. Now, the end of time, through which we used to exit to the conceptual level, will be coincident with the end (i.e., the termination) of statistics altogether, and with the end (the termination) of the time series altogether. Surely, the volcanic activity of trading the underlying asset was summarized by a volatility σ, which was only a symbol, which was constant because it was a symbol and never implied that the description of the abyss was really Brownian motion with constant volatility. To the contrary, we said, anything could still happen in that model – any event – and Brownian motion was only a model,

God's model. Yet, Brownian motion was a choice among many – as a matter of fact, the simplest one. That it develops in time formally could not but introduce the ambiguity of time and the ambiguous thought that other choices of temporal processes, which could have competed with it conceptually, will actually compete with it in time, before the end of time. As we do not need to sample Brownian motion in finite time and there is truly a sense that we can make of instantaneous volatility (its estimation converges instantly, whereas the estimation of drift, for example, doesn't), the temptation is great, while we are at the summary of the volcanic activity anyway, to summarize it with a more complex process. As volatility can be instantly defined and even estimated, we reason that, perhaps, we shouldn't consider the end of time but consider instead, immediately and on the spot, all the other complications. Who could stop us? And which one to adopt? Yet, now, with the event of writing the contingent claims, we have the possibility, even the obligation, to adopt none. Now, truly an alternative summary of the volcanic activity can be envisaged, in which the event is fully present, and no longer maintained by the power of thought. Now, the trading activity, in its fully volcanic character, and the event, in its full abyss, can be represented, and even materially had. They can be had outside time, which was always perplexing, and in place, instead.

Let me try to rephrase this. A trading pit is no ordinary place; it is not a random generator. It may look like one from outside, once you exit its volcanic time and activity and consider the mere generation of numbers. It is true that the trading pit only produces prices, which look like numbers, but what it achieves, chiefly, is to be in constant contact with the event. The event is always negatively expressed, relatively to any frame of reference that tries to capture it or represent it. For the players engaged in a game of dice, the event is not any of the outcomes; it is that a meteorite falls on their head and kills both, thus ending the game. For the random walk or for Brownian motion, the event is not that the price should go up one tick or down one tick; it is that the price should jump all of a sudden, or that the volatility should jump all of a sudden. If the trading activity is to be represented in time, as a stochastic process, the event will always be, in time, that the stochastic process that was chosen for the representation was not the right one, because it did not account for that extra event. No matter how complex or complete the stochastic process we may have selected, it will itself exactly delineate the event of its own failure. It is impossible to write in advance the stochastic process that would take into account the possibility of all events. We cannot write the probability distribution in which any probability distribution would occur. This is because a stochastic process or a probability distribution is inevitably quantitative, and quantity is inevitably contradicted by another quantity. This is because of time and of the necessity to wait, in time, until after the event, in order to produce the backward narrative which, as Bergson (1946) writes, produces the possibilities of the event. This, as we said, leaves then no choice for thought but to struggle hard in order to maintain that a given stochastic process that it has picked as a representation of the volcanic activity, although unfolding in time, in reality is produced after the end of time, after all the events have been recorded – in other words, to maintain that a quantity is, in reality, a quality. This makes us wonder whether the record of the end of time couldn't be had and represented in another dimension than time, if only in order to prevent the ambiguity and the continual backward narrative.

This, we believe, can be achieved by the chaining and the repetition of tokens of the event that we have described above – namely, the event of writing and trading the contingent claim after the contingent payoff has been manufactured. In other words, it can be achieved by the price series of those contingent claims of increasing complexity, instead of a time series of the underlying asset price. When the market-maker of contingent claims steps back inside the trading pit (and not to its side) in the illegitimate move that transforms a conceptual certainty (that of the premium to manufacture the contingent payoff) into the making of a price (for the contingent claim), he produces an event that cannot be reprocessed in the previous frame by a backward narrative. The newly written contingent claim was not part of the previous world or the previous time or the previous narrative. It takes nothing less than the certainty of the conceptual exit from the trading pit of the underlying asset to be capable (a capacity, not a possibility) to *write* a derivative and to make its market back into the pit. Its written matter, its materialization, is of the order of that certainty; hence, it doesn't mix again with the object level in which the trading of the underlying asset was described.

The invention of the writing of the contingent claim is not a criticism, say, of Brownian motion, to the effect that volatility is uncertain or stochastic or that there exist jumps. It is generated from the void outside the formalism, from the very 'space' in which the ascent to the conceptual level is measured. The invention of writing is an event of incommensurable magnitude. Perhaps the move to conceptualize the event and the volcanic activity of the trading pit wasn't legitimate in the first place. If the world is reduced to the view from inside the pit of the underlying asset, and if time is only ever volcanic, one not only wonders how the imagination of statistics and the access to time series could be possible, but also how any derivative could be written at all. To imagine writing the derivative presupposes

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a preoccupation with time that is incompatible with the volcanic time of the trading pit. One has to take a step back to imagine writing the derivative. On the other hand, there is now pressure that we shouldn't have left the trading pit anyway, even in a move to conceptualize the event, because the event is so abysmal and so absorbing. So, perhaps one worry could fuse with the other, and we could argue that the only possible 'conceptualization' of the event is not actually a conceptualization but a materialization. As we exit the pit conceptually we at once re-enter it materially with the newly written contingent claim, whose written matter becomes now equivalent to the certainty of the concept of trading – so, in this sense, it is obtained *through* the force of trading and is not a step back – yet, is, at the same time, literally its transmutation. There is truly no outside view. The event is unconceivable and should never be conceived. The medium of the event is immanent and material. We are materially in the middle of the event. Thus, the event was never conceptualized – it becomes material.

This whole story, which I have narrated about the event and about the virtual chain of prices of contingent claims, is how I think the market should be thought of

The matter of the event

There is a malediction associated with numbers, which is that prices will always look like numbers, once you withdraw them from their true genesis, and that numbers, once given in time, can only serialize as time series. A price is attached to a contingent claim by the act of an exchange. In *The Medium of Contingency* (2015), I argue that this is no slight matter and that prices are not just numbers. They are not values. They are attached to a material sheet of paper, the contingent claim, not to a probabilistic state. As such, the contingent claim has the event, the abyss, the end of the world written all over against it, simply as the other side of the paper it is written on. The call option returns a different payoff, depending on whether the price of the underlying asset rests above or below the strike price at expiration – its payoff is said to be contingent upon those states. However, the call option, or any contingent claim, really, is also, and perhaps above all, contingent upon the continued existence of the market at expiry, the continued existence of money, the continued existence of the whole world, or, to put it more simply, the continued existence of the very sheet of paper it is written upon. There wouldn't have been an exchange, therefore a price, without the materiality of the contingent claim, whose other face is the possibility of disappearance of that matter. In a sense, price has already the end of times factored in its genesis.

When we talked, earlier, of the trading pit and of the randomness of the price, we imagined an ongoing process. We didn't trouble ourselves with price being the result of an exchange, or worry about the abyss over which it was, by its very genesis, already stretched. We immediately considered the next movement of price. Apart from the eventuality that the end of the world might actually happen right in that instant, there was nothing, in the next movement of price, to materialize the *virtuality* of the end of times and to let us feel that price is, in a way, already waged on the end of times. But now, there is the whole chain of prices of contingent claims, which has materialized thanks to the interpretation of the real event that BSM represents. A simple thing is an infinite thing, writes Bergson (1902). Once we understand that BSM is *simply the consummation of the concept of the market of the underlying asset* – the volatility of its price and its concomitant self-financing trading in variable size – and that the certainty of that concept (a semantic certainty, we said) is translated into the writing of contingent claims and into *their* subsequent trading; once we understand that the event that the market of the underlying asset constitutes is always already translated into a trading event of higher order, which falls in the same pit back again, or that the 'absolute value' of price becomes a price in turn, *through* the certainty of that absolute value and the certainty of that concept and not on the side of price, we see the matter constituted before our very eyes, the matter that the event is made of.

Once we understand that the price of the underlying asset is no longer followed by time, or by volatility, or by any conceptual exit under the form of a stochastic process and such that the event would make us resort to the backward narrative, but is followed by the price of the derivative written on it – better, by the prices of all the derivatives written on it, in a variation which is already, by itself and before time passes, a total variation and a total 'volatility smile' (because the writing, let alone the pricing, of contingent claims was not conceivable before and because it took the ascent in the conceptual level to materialize it), we realize that the process leading to the event no longer occurs in time or varies in time, but becomes at once materially present in the pit.

The event is the very matter that the market is made of. There never was probability in the market. There has always been the event. Probability was only considered in the formalism, in Brownian motion, then in BSM. However, the right interpretation puts us back in contact with matter, with the invention of the writing and of the trading of the contingent claims. The statistical exit and the whole algorithmic trading project that issues from it have always been the wrong route. Even more radically, any statistical framing or conception of the market is a wrong route. We persist in the distinction between the two time registers, and we now argue that the idea of a stochastic process was only ever meant as a representation of trading and of the volcanic time of the pit, as a translation of the meaning of the market and of price (quantitative, for lack of another choice), not as a model of the actual time series of prices.

Here is our claim: Writing a stochastic process for the underlying price was only ever meant as the preamble of BSM and its formal derivation, the first exit from which (into reality) is not the volatility σ but the traded price of the contingent claim directly. Better, the first exit is the whole collection of traded

prices of all contingent claims, and this disqualifies any written stochastic process, no matter how general, as an attempt to exit. We don't know what CAPM is up to, with its handling of volatilities of asset prices and its whole notion of the time register in which all these statistics occur. Of course, there exist empirical time series of asset prices, but now we are saying that the only way to grasp them in thought, the only way to project their past record forward (for thinking is always thinking forward), is through the prices of contingent claims written on those assets, which are not actually conceptions but materializations, as they do nothing but extend the market and continue the trading.

This whole story, which I have narrated about the event and about the virtual chain of prices of contingent claims, is how I think the market should be thought of. I had to get the story right, because it is virtual and because it is the story of the event. The event is virtual; it is not actual. Statistics can only gather actualities. No statistics can represent the abyss of the event. Only the market can, provided the story is told right. The story needs the layer of the formalism as part of its matter. The event is truly material. Contingency is absolute and has to be thought of independently of the frame of reference of possible states and probability. As such, it is material because matter is absolute. Possibility and probability are immaterial. A new metaphysics has probably to be invented in order to recognize how material the event and the market are; and concomitantly, how different they are from probability.

Money and finance are key in the definition of probability. If anything, money is the ground, not probability. De Finetti (1974) defines subjective probability as the odds that a banker quotes for you, to bet on the corresponding event. Thus, subjective probability is down to a financial transaction. As for objective or frequentist probability, it is defined, by von Mises (1981), as the limiting frequency of a certain outcome in a sequence of random outcomes; and a sequence of outcomes is said to be random when it is invulnerable to gambling strategies. Hence, at bottom, a financial argument, once again. Having recognized money as the fundamental category behind randomness – a category even more fundamental than time, for we no longer count in time or expect frequency in time; we count in money – Shafer & Vovk (2001) propose, for instance, a complete reformulation of probability theory with money and capital processes at its basis, rather than measure theory.

This is a reformulation of probability theory at large, not in its specific application to finance. Trading is more fundamental than probability and doesn't need probability to frame its randomness. Enough to consider the efficient market hypothesis and the conclusion will be that the path of price has to be random at any time scale – which just gives Brownian motion as the simplest case. This *absolute value* of the market of a certain asset, as I have called it above, is translated into the price of the contingent claim written on that asset, then the randomness of price of the latter, which is now traded, is translated into the price of the next, so on and so forth, in an argument which I have tried my best to transplant from time into place, from the metric of probability space into the topology of the event. Money and writing are the only reason why this is so, because they entail trading. They are present both at the beginning and at the end. No event disrupts the market from outside; the market itself is the continual event. This is the reason why, contrary to probability, it is commensurate with the event.

ENDNOTE

1. By register, we mean the form and compass of a certain discourse, as used in particular circumstances.

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