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The End of Probability

Not only does the market not need probability, the market replaces probability

Contingency

Probability theory and the metaphysical category of possibility are based on the notion of "states of the world" (or possible worlds). In the market, the only states of the world are prices. Contingency is a very general category that is independent of the later division of the world into identifiable states or the recognition of the different possible worlds that the world might be. Metaphysical thought *later* works contingency into the notion of separable possible states. However, pure and absolute (and initial) contingency only minimally says that the world or that the things could have been different.

A contingent claim is generally a claim that pays out something, but that could pay something different (following the minimal definition of the contingent world as a world that is so but that could be different). Because the only thing that can make a difference in the market is prices, and because different prices are the only thing that can suggest that the actual world could have been different, we usually define the difference of the contingent claim in terms of underlying prices: if S is greater than K, pay \$1; else, pay 0 (where S is the price of the underlying of the contingent claim, typically the price of an underlying share).

In the real world, contingency is not reducible to underlying states, because truly unpredictable events are typically those that escape the previously known range of possibilities or possible states. The current saying is that true events create the possibilities that will have led to them.

It is out of the question to apply probability to those events, or even to say that their probability was zero, because probability is only defined relative to a stable and well-defined universe of possibilities.

It is our metaphysical tendency to reduce contingency to identifiable possible states that makes our models and view of the world vulnerable to their outside, or to contingencies that were not previously identified. Consequently we ask: Why not drop the division into states (i.e., possibility) altogether and deal with contingency pure and simple (i.e., without states)?

The market

Possible states and probability theory are pervasive in the market because we feel confident that the underlying states there are prices anyway, so the range of possibilities seems to be identified and totalized once and for all. What radical change could contingency bring over and above prices? No matter how many unpredictable events hit the market, the "world" of the underlying share would always consist of the array of prices of that share, wouldn't it? Because the market is composed of numbers (prices), we feel confident applying probability to it.

However, precisely because of the precision of numbers, the market is precisely the domain where probability and states of the world will demonstrably fail, in front of the category of price which, in my view, is supposed to replace probability and states of the world.

Indeed, if states of the world in the market are prices, then the prices of the contingent claims should also be states of the world, different from those of the underlying. This is what probability theory, and its culmination – which is derivative valuation theory – cannot allow. According to derivative valuation theory, the derivative value is a deterministic function of the underlying price. In the framework of Black–Scholes–Merton, option value is a deterministic function of the underlying and options are redundant. However, everybody knows that options trade independently of their underlying in a market of their own, thus adding new states of the world (in other words, trading options turns the BSM volatility stochastic). The purpose of writing any contingent claim is to trade it and to add a new price to the market. Yet, the framework of fixed states of the world and of probability automatically leads to the dynamic replication of the contingent claim and hence to its redundancy.

The only way out is to conceive of contingent claims and their market prices independently of the whole framework of possibility and underlying states of the world. Surely, the price of a contingent claim only depends on its underlying at maturity, and this is the only reason why we call it a "derivative." This is the only reason why derivative valuation theory, which is just the mathematical exploration of the terminal dependence of the contingent claim on its underlying, likewise calls it a "derivative." However, the market of the contingent claim is what happens *before* its maturity. (Derivative valuation theory ignores the market of contingent claims.) Before it expires, the price of the contingent claim may depend, not only on its underlying, but also on the volatility of the underlying, on the volatility of its volatility, etc. In a word, it may depend on the whole market.

How we express this instant nesting (or "complication") in the language of prices is by saying that vanilla options never trade alone, like their valuation theory prescribes. Barrier options, variance swaps, options on variance (options on VIX), cliquets, etc. – all these structures trade as well at prices not necessarily predicted by the model that we had initially, candidly, thought was all we needed in order to value the vanillas, i.e., BSM. As a consequence, no one can claim to rightly price vanillas unless one's model is calibrated to the market prices of barrier options, variance swaps, options on variance, etc. The rule is constant recalibration of the model to the novel prices of novel structures. This is the rule of the market. The market is a constant Black Swan. It constantly breaks any previously defined range of possibilities.

The tree of possibilities

We should forget about probability and stochastic processes altogether. Every day brings a new market, to which we recalibrate. Probability and stochastic process impose on us the notion of a transition. We move from one day to the next by transitioning between the given states of the world, with some probability. In reality, however, the whole idea of a transition should be severed. There is no such thing as a tree of possibilities, and consequently no transition. The market attaches prices to the contingent claims immanently, without a supervisor, without the transcendent view of states of the worlds which are supposed to distribute probability once and for all.

If the contingent claim is thought of as a derivative, i.e., as a function of its underlying, this imposes on us that we chart the corresponding underlying states. Not only do we have to identify them, but by the same token, we eliminate all the others: we reduce the contingency of the world to only those states. By contrast, if the contingent claim is only conceived as the written formula that it is (pay \$1 if S is greater than K, 0 otherwise), its only underlying will be the sheet of paper on which it is written. This sheet is not divided into states (those above K and those below) and doesn't presuppose the notion of states. It doesn't impose on the world that the only states that it might experience the next day are states of the underlying S. For all we know, nuclear war might break out the next day, or the world as a whole might disappear. The sheet of paper is indifferent to all of this. It will still exist the next day and it will still admit of a price (provided the market still exists). By contrast, the probability of a state of the world that was identified the day before may no longer be defined the next day, because of the major shift of the whole range of possibilities.

Recognizing absolute contingency is only recognizing that contingency should be considered absolutely and not derivatively on identifiable possibilities. Time passes every day and the world is still contingent the next day, without there being an array of states of the world that mediate the transition from one day to the next. Contingent claims admit of prices one day, and they admit of prices the next day, without there being any common measure between what we

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identify as "states of the world" on one day and "states of the world" on the next day.

Valuation theory shackles us to the tree of possibilities, with the necessity that the states of the world of the next day be commensurate with those of the previous day because the theoretical value of the contingent claim is computed as a backward induction in the tree. We need to project all future possible states (as if the future was perfectly imaginable and wasn't absolutely

incommensurate with the present) in order to compute the present value of the contingent claim as the discounted mathematical expectation of its value in the future states.

In reality, however, the market prices the contingent claims going forward. Each day brings a new price and a new market. Valuation theory seems to know of no other way but backwards. Instead of forcing the market into valuation theory and thinking of the market price as the theoretical result of some general equilibrium problem-solving algorithm, no less based on fixed and identifiable states of the world, why don't we just embrace the market as a radical alternative to valuation theory? We just have to admit that the market finds the price of the contingent claim, immediately and immanently, by definition of the market.

Writing and exchange

Contingency is absolute; it doesn't need the states of possibilities, or the tree of possibilities, or the transitions (all these constructs are too fragile and too unstable). Every day the world is the way it is, and every day it could have been different. To repeat, there is no visible or identifiable transition. As a matter of fact, we go to sleep in the interval and nobody knows what happens when we sleep! Contingency is indifferent to the passage of time. There should be no difference between a thing that exists already or is even past yet could have been different, and a future thing that doesn't exist, yet we know is contingent. Contingency comes before existence. By contrast, possibility is sensitive to the passage of time. A past thing is no longer possible, because it becomes actual.

In the market there are only contingent claims. It is only incidentally that the contingency of a certain contingent claim may be defined relative to the prices of another (which is then called the underlying). As I have said, this is due to the fact that prices are chiefly what distinguish between different worlds. However, nothing stops us from defining contingent claims whose underlying may be the weather, or earthquakes, or corporate defaults, etc.

The market is the medium of contingent claims, and it translates them into prices. As I

have said, contingent claims are recognizable by their written formula. It is because they are written that they can dispense with possibility and its underlying states of the world. Even though the written formula may have a hundred provisions, it will remain ONE written formula, which will be interpreted the next day, and the day after the next, etc., and it will admit of a price every day. At no point do we need to decompose it into states.

Now the other side of something being written is that this something will then be exchanged. It is not a coincidence if writing enables us to collect the contingent claim in one undivided formula and if, on the other hand, it lends itself to the exchange. It is the same matter (the material sheet) on which the formula is written which is subsequently exchanged. This might even act as the definition of writing: something that collects the difference of the contingent claim on one side and for this reason admits of a price (i.e., is exchanged) on the other side.

Probability theory thinks only of the stochastic process of the underlying (i.e., underlying states and the corresponding probabilistic transitions) and the value of the derivative is then tracked by stochastic control (i.e., dynamic replication or dynamic programming). However, all of this is not real. What really exist are contingent claims, their market, and their prices. Nobody ever observes a stochastic process. We are not even sure what probability is! The only pricing technology that is worth having is a tool that can be calibrated and recalibrated to the market prices, without any presumption of states. The only reason why probability and stochastic control are episodically needed is to insert the dynamic trader in the process. He has to compute hedging ratios and value derivatives relative to other derivatives only in order to remain afloat in the market and to earn the right to recalibrate the tool the next day.

The market as a category of thought

Of course my whole discourse presupposes the existence of the market. I won't call this a "theory" because contingent claims exist in practice, not in theory, and so does the market. As a

matter of fact, I am trying to avoid theory as much as I can!

You observe that my entire "theory" presupposes the existence of the market and you fear this might be a weakness, as if the market was too special or too fragile and as if, by contrast, probability was something altogether more general and more established. You argue that the market is just a social phenomenon, as such relative and situated, whereas probability is abstract, pure, and metaphysical.

My whole point is precisely to argue to the contrary. Indeed, why wouldn't probability itself be considered a social construct? (There are even philosophers who argue that scientific theories as a whole or even reality, as a metaphysical concept, are just social constructs.) Why would probability be more general, more abstract, and somehow purer than the market? Conversely, why wouldn't the market be considered metaphysical, i.e., a pure category of thought, just like probability?

Simply define the market as the place where contingent claims get prices attached to them. Why would such a place sound stranger and more "improbable" than the "place" where states of the world are assigned numbers that we call their "probabilities"?

Who assigns probabilities anyway, and what does probability mean as a matter of fact? Ironically, probability is in fact philosophically defined after price. Subjective probability is defined by de Finetti as the odds that a "banker" is supposed to quote for you in order to bet on the outcome of a certain event, whose probability we will just define as these odds. As for objective probability, it is defined by von Mises as the limiting frequency of the occurrence of the event whose probability we wish to define within a perfectly random sequence, or as the fair price of a lottery in the long run. The infinite "random sequence" or the "long run" in question are supposed to be truly random, i.e., unbiased, and "true randomness" is then defined as a sequence of outcomes that are insensitive to gambling systems. "Banker" and "gambler": precisely the personae who deal with money and prices, not with probabilities.

The market as a substitute to probability

Still, the market and the notion of price may strike us as depending too much on human beings, i.e., on two partners exchanging the contingent claim at that price, while probability is just independently found in nature, attaching to the thing in itself. Well, is it really? Nobody can observe probability. We only observe statistical regularities.

Statistical laws are empirical laws. However, to postulate a random generator, or a probability for the single case (what Popper calls "propensity"), is a metaphysical move, what philosophers call a reification. You can think of it as a shortcut. Instead of thinking of the whole statistical population exhibiting the given distribution, we postulate a random generator that will generate each individual in turn, under a probability distribution bearing the same moments as those we have inferred statistically from the population. However, nothing grants this metaphysical extrapolation. We shouldn't forget Nassim Taleb's criticism at this juncture. There is no finite amount of statistical observations that can permit us to pin down the probability distribution that we have assumed exists behind the scenes. We really have to postulate one.

The philosophical concept of probability may have never come into existence. Ian Hacking dates its emergence back to the seventeenth century (gambling, dice, etc.) and its consolidation to the nineteenth century (statistics). Mathematical probability theory by Kolmogorov is in fact only measure theory, bearing on set theory. It certainly has axiomatized probability calculus but it has nothing to say about "physical" probability.

Stochastic processes are very well defined mathematically. They are the expression of probability theory at its finest. But what pure stochastic processes do we really know to exist physically? Brownian motion (of the pollen particle) is just the phenomenological summary of a multitude of invisible shocks occurring between the particle and the molecules of the liquid it is immersed in. It is not pure. It is not a real stochastic process. As for the Brownian motion of

market prices, it is no less the summary of a multitude of minute causes and transactions.

I seriously ask why the material couple formed by contingent claim and price cannot replace the metaphysical couple formed by state of the world (i.e., possibility) and probability. It is not probability that inspired us to write contingent claims in ever more complex shapes (as you seem to suggest). Whoever wrote the nth complex contingent claim must have just had in mind the complex condition where it would pay off. He never thought of probability. He only thought of writing this payoff.

On the contrary, it seems to me that whoever thinks of a complex (abstract) state of the world and tries to figure out its probability is in fact somebody who just stops short of writing the corresponding contingent claim. My advice to him: just write down this complex condition;

kernel" exists. This means that the prices of contingent claims that are written on an underlying have to be expressed as the discounted mathematical expectation of their payoff under a certain probability distribution of the underlying (a.k.a. risk-neutral probability distribution). This doesn't mean that the underlying prices are in effect generated by this probability distribution, or that probability even exists! All that it means is that the pricing operator should be positive and linear. As a matter of fact, when probability itself is defined after price (as we saw with de Finetti), it is specified that the famous "banker" quoting those odds has to make sure he is not arbitraged away!

In other words, it is the market prices of the contingent claims that "generate" the prices of their underlying, not the other way round. What I mean by this is just the observation of what the

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just materialize it in writing; get it out of your head! Then my whole idea is that, as he in effect writes it down, the material sheet on which he writes it is *ipso facto* meant to be exchanged in a market.

Don't ask why or how. This is both the definition of the market and the definition of writing. One always writes for somebody else (one writes a letter, or a testament, etc.) When you write something, you let go of it, you let it float; and this just means that the written contingent claim will admit of a price. Contingency is written *therefore* it is exchanged. Both the writing and the exchange are material. Possibility, by contrast, is immaterial, and so is probability.

Non-existence of the random generator

The prices of contingent claims have to verify the non-arbitrage principle. A theorem states that non-arbitrage is enforced as soon as a "pricing

market-makers of contingent claims do every day when they use their pricing models. They calibrate them to the prices of vanilla options, barrier options, variance options, etc., and they infer this famous risk-neutral probability. The next day they recalibrate, possibly enlarging the pricing model in order to accommodate the prices of more complex contingent claims. The risk-neutral distribution they infer thus keeps shifting. This disrupts the whole notion of a random generator, which would stably generate the prices of the underlying.

As a matter of fact, the existence of a market of contingent claims, where none should be redundant and all must independently trade, is a direct proof of the non-existence of a random generator for the underlying. In other words, it is a direct proof of the non-existence of states of the world. The whole metaphysical notion of possibility has to go away, together with probability.

The step beyond

A further objection to my "theory" is that the market price can replace probability only in the market-specific situation. How could the market replace probability when dealing with the prob-

sure Deleuze would agree corresponds to what I call "recalibration").

All I am saying, in the end, is that, because the market price is the translation of the contingent claims without the intermediary of states

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abilities of events other than the triggering of payoffs of contingent claims?

My objection to the objection: Is it really probability that is applied to those "foreign" events? Surely probability can be applied to dice, to roulette wheels, to the motion of the pollen particle, to population statistics, etc., because this is circular. Indeed, these are precisely the statistical phenomena from which probability emerged as a concept to begin with. However, the real challenge for probability is to apply to real events (happening outside the casino, or the tables of actuarial science), to singular events that are in no ways statistical. And what are "real events"? Precisely events that disturb the range of possibilities on which probability was supposed to be defined – what Taleb calls "Black Swans."

The whole charge that Taleb is mounting against probability and its theorists is precisely concerned with such events. Badiou is the main philosopher of the event who formalized, more than 20 years ago, all that Taleb is trying to say about Black Swans. At no point does Badiou mention probability. Deleuze is the other, perhaps less formalistic, philosopher of the event. He speaks of the Nietzschean dice-throw (which I am sure Deleuze would agree is the market of contingent claims, if only he had known it like I do) and of the "empty square" that keeps redistributing the probability distributions (and which I am

of the world and their artificial delimitation – and because the whole trick of writing the formulas of contingency over the contingent claims amounts to getting rid of the underlying states – the market and price may just be the substitute of probability that is needed in such situations! True, the task remains to see how the notion of price can possibly be generalized to domains other than financial.

I think we must first try to generalize the idea of the writing of contingent claims. I believe the lead lies in writing. What written stuff people exchange when dealing with events that are more grandiose than financial payoffs are books. And the corresponding market is the sphere of thought at large. The only problem is that books are priceless.

Going forward

In conclusion, I am really, seriously arguing for a direct passage from contingent claims (i.e., material writing instead of immaterial possibilities) to prices without ever mentioning probability. Surely probability can help us statistically analyze past prices, i.e., infer probability distributions from the observed statistical regularities. However, this is circular, as I have said, because "objective probability" is just another word for the statistical regularity that the metaphysicist dreams of elevating to the status of law of nature. However, probability can be of no help

with regard to future prices, as random generators simply don't exist in the market. Therefore, probability can in no way help us make educated guesses.

The probabilistic tools that we use in the pricing of contingent claims are always used in reverse. That is to say, their only use is to calibrate them to the market prices of contingent claims and to thus infer the risk-neutral probability which will allow us to price other stuff without creating arbitrage opportunities. In other words, the tool is just a sophisticated interpolator/extrapolator of prices of contingent claims.

Finally, a word about how this duality between price and probability, or between forward and backward, has first struck me. (Probability is the backward view, of course, and price is the forward view.) Actually, this duality is well known to quants who deal with the so-called backward and forward pricing equations.

A backward partial differential equation (typically the Black–Scholes equation) produces the price of a single call option (say) for all times t lying between its maturity and the present time and for all spot levels S. It just computes the discounted mathematical expectation of its payoff in all those states of the world (S, t).

By contrast, a forward equation produces the prices of all call options of different maturities T and different strikes K as seen from the present spot and present time, i.e., in the present market. My contention is that the forward equation, although based on an algorithm and a mathematical formalism that are just the reciprocal image of the backward equation, is not a computation of mathematical expectations. In fact, it is not based on probability. Its state variables are not states of the world (S, t) but precisely the marks (K, T) that are written on the contingent claims, namely their strike and maturity. The forward equation is the instant view of the instant market. You have to use a different forward equation if you wish to compute the call prices from a different spot and different time, i.e., in a different market situation. There is nothing to stop the call prices from moving completely unpredictably between two computations. As a matter of fact, forward equations are better suited for calibration and recalibration. W