Pricing Cross-Currency Convertible Bonds with PDE

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Abstract
In a finite element framework, we analyze the pricing of cross-currency convertible bonds where the underlying share is denominated in a currency foreign to the convertible bond issue. Especially of interest are the cases where this two-dimensional problem cannot be reduced to one dimension.

Keywords : Cross-currency convertible bond, PDE, linear complementarity problem, finite element method.

1 Introduction

Convertible bonds are becoming increasingly popular instruments among arbitrageurs, because of their equity-to-credit hybrid nature. They are sensitive both to the volatility of the underlying equity and to the volatility of the issuer's credit. Their arbitrage-free pricing equation is difficult to solve, not only because of the complexity of the underlying theoretical model, but also because of the complexity of the instrument itself. Indeed, even when the pricing problem is formulated in a one-dimensional framework, where the underlying equity is the only state variable, the interest rate is a deterministic function of time and credit is a deterministic function of time and equity level, we have to resort to complex numerical algorithms because of the multiple embedded options, inherent in the convertible bond.

Beside the option to convert the convertible bond into shares of the issuing company, such embedded options may include the option, for the issuer, to redeem the bond before maturity (a. k. a. issuer's call) for a specified redemption price (a. k. a. call price), and the option, for the holder, to put back the bond at a pre-specified put price. We refer the reader to Ayache, Forsyth and Vetzal (2003) for the analysis of the pricing equation and its numerical solving techniques in one-dimension.

A further complication occurs when the convertible bond is a cross-currency instrument, or in other words, when the underlying share is denominated in a currency foreign to the convertible bond issue. Strictly speaking, the valuation becomes a two-dimensional problem as both the process of the underlying share and the exchange rate have to be modeled. In most cases, however, we can reduce the problem to one dimension, by assuming as new variable the value of the equity in the domestic currency. A reason why we may not always do so is the presence of an issuer's soft call, that is to say, a right to redeem the bond in advance that is triggered only if the underlying share trades above a certain trigger level. As the trigger is expressed in the currency of the share, this necessitates the separate modeling of the share process in the foreign currency, therefore imposes the two-dimensional framework.

We propose a finite element scheme to solve this two-dimensional problem. The goal being accuracy and speed, we emphasize the construction of an adaptive mesh. Like we said, the convertible bond is a complex instrument, and our two-dimensional solver is but one instance of the general two-dimensional equations and solving routines that we may wish to apply to its valuation. Alternative two-dimensional problems may involve the interest rate as the second state variable (see Yigitbasioglu (2001) and Zvan, Forsyth and Vetzal (1999)), or the hazard rate.

2 Definitions and notations

2.1 Convertible bond

The convertible bond is a bond that can be converted into shares. The conversion ratio R is stipulated in the contract and may depend on time. In addition to the holder's right of conversion, put and call provisions are common.

Call provision. The issuer has the right to redeem the convertible bond, prior to maturity, against payment of the call price K_c. When the bond is called however, the right of conversion of the holder prevails. Both R and K_c may depend on time. There are two kinds of call provision.

- Hard call. The issuer can call back the convertible bond unconditionally (the holder may then exercise his right of conversion).
- Soft call. The issuer can call the convertible bond conditionally on the share being above a trigger T.

Put provision. The holder has the right to sell back the convertible bond to the issuer at a specified strike price K_p prior to maturity. K_p may depend on time and $K_p < K_c$.

2.2 Cross-currency case

We denote S the underlying share value, V the convertible bond price, t_0 the initial time, t the current time, and T_f the maturity. Without loss of generality, we subsequently assume that both R and K_c are positive constants.

We study the cross-currency convertible bond where the underlying share is denominated in a currency foreign to the convertible bond issue. We choose the currency of the convertible bond as the base currency, and the currency of the underlying stock as the foreign currency, and we let X be the foreign exchange rate, that is the number of base currency needed to purchase one unit of the foreign currency. According to these definitions, the value of the stock is S in the foreign currency and SX in the base currency. The price of the convertible bond is then clearly a function of both the underlying price S and the exchange rate X. We remark that in the cross-currency case, K_p and K_c are expressed in the base currency while the trigger T, like S, is expressed in the foreign currency. Note that R is a number of stocks which does not depend on currencies.

We assume that S and X follow standards log normal diffusions

$$
\begin{aligned}
dS &= \mu S dt + \sigma S dW, \\
dX &= (r - r_f) X dt + \sigma_X X dW^X,
\end{aligned}
$$

where W and W^X are two standard Brownian motions with a correlation coefficient ρ, σ and μ are respectively the volatility and the drift of the share, σ_X is the exchange rate volatility. r is the domestic risk-free interest rate (for the base currency of the bond) while r_f is the foreign risk-free interest rate (for the currency of the share).

We define finally $GP \equiv RSX$ the gross parity and F the redemption value.

Our goal is to price such a convertible bond.

3 Preliminary analysis

We seek to write the convertible bond pricing problem as a Linear Complementarity Problem (LCP) (see Paul Wilmott, Jeff Dewynne and Sam Howison (1993) and Ayache, Forsyth and Vetzal (2002)). It is the precise mathematical formulation of the following intuition. Let

$$
\begin{aligned}
\mathcal{L}_0 V \equiv\ & \frac{\partial V}{\partial t} + \frac{1}{2}\sigma_X^2 X^2 \frac{\partial^2 V}{\partial X^2} + \rho\sigma\sigma_X XS \frac{\partial^2 V}{\partial X \partial S} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} \\
& + (r - r_f) X \frac{\partial V}{\partial X} + (r_f - \rho\sigma\sigma_X) S \frac{\partial V}{\partial S} - rV.
\end{aligned}
\tag{1}
$$

The value of the convertible bond is given by the solution to $\mathcal{L}_0 V = 0$, subject to the constraints, in the soft callable general case,

$$
(\mathcal{C}_0)\ \begin{cases}
\text{If } S \geq T \text{ and } V(S,X,t) \geq K_c \\
\quad \text{then } V(S,X,t) = K_c. \\
\text{If } V(S,X,t) \leq GP \\
\quad \text{then } V(S,X,t) = GP. \\
\text{If } V(S,X,t) \leq K_p \\
\quad \text{then } V(S,X,t) = K_p.
\end{cases}
$$

Non callable convertible bond. V solves the LCP

$$
\begin{pmatrix} \mathcal{L}_0 V = 0 \\ V - \max(K_p, GP) \geq 0 \end{pmatrix} \vee \begin{pmatrix} \mathcal{L}_0 V \leq 0 \\ V - \max(K_p, GP) = 0 \end{pmatrix}
\tag{2}
$$

where the notation $(f = 0) \vee (g = 0)$ is to be interpreted as at least one of the equalities $(f = 0)$ or $(g = 0)$ holds at each point.

Hard callable convertible bond. There are two cases:

- $GP \geq K_c$. The convertible value is simply $V = GP$ since the holder would choose to convert immediately.
- $GP < K_c$. In this case, the LCP writes

$$
\begin{aligned}
&\begin{pmatrix} \mathcal{L}_0 V = 0 \\ V - \max(K_p, GP) \geq 0 \\ V - K_c \leq 0 \end{pmatrix} \vee \begin{pmatrix} \mathcal{L}_0 V \leq 0 \\ V - \max(K_p, GP) = 0 \\ V - K_c < 0 \end{pmatrix} \\
&\qquad \vee \begin{pmatrix} \mathcal{L}_0 V \geq 0 \\ V - \max(K_p, GP) \geq 0 \\ V - K_c = 0 \end{pmatrix}
\end{aligned}
\tag{3}
$$

Soft callable convertible bond. There are two cases:

- $S \geq T$
  - ◇ $GP \geq K_c$. The convertible value is simply $V = GP$ since the holder would choose to convert immediately.

- $\diamond$ $GP < K_c$. The LCP writes then

$$\begin{pmatrix} \mathcal{L}_0 V = 0 \\ V - \max(K_p, GP) \geq 0 \\ V - K_c \leq 0 \end{pmatrix} \vee \begin{pmatrix} \mathcal{L}_0 V \leq 0 \\ V - \max(K_p, GP) = 0 \\ V - K_c < 0 \end{pmatrix} \vee \begin{pmatrix} \mathcal{L}_0 V \geq 0 \\ V - \max(K_p, GP) \geq 0 \\ V - K_c = 0 \end{pmatrix} \tag{4}$$

- $S < T$

$$\begin{pmatrix} \mathcal{L}_0 V = 0 \\ V - \max(K_p, GP) \geq 0 \end{pmatrix} \vee \begin{pmatrix} \mathcal{L}_0 V \leq 0 \\ V - \max(K_p, GP) = 0 \end{pmatrix} \tag{5}$$

In all cases, the terminal condition is given by

$$V(S, X, T_f) = \max(F, GP). \tag{6}$$

3.1 Change of variables

Using the change of variables $Z = \ln(S)$, $Y = \ln(X)$ and time inversion $t^* = (T_f - t)$, $\mathcal{L}_0$ becomes a new operator $\mathcal{L}$ defined by

$$\begin{aligned} \mathcal{L}V \equiv & -\frac{\partial V}{\partial t^*} + \frac{1}{2}\sigma_X^2 \frac{\partial^2 V}{\partial Y^2} + \rho\sigma\sigma_X \frac{\partial^2 V}{\partial Z \partial Y} + \frac{1}{2}\sigma^2 \frac{\partial^2 V}{\partial Z^2} \\ & + \alpha(T_f - t^*)\frac{\partial V}{\partial Y} + \beta(T_f - t^*)\frac{\partial V}{\partial Z} - r(T_f - t^*)V, \end{aligned} \tag{7}$$

where $\alpha(t) = r(t) - r_f(t) - (\sigma_X^2/2)$, and $\beta(t) = r_f(t) - \rho\sigma\sigma_x - (\sigma^2/2)$. The gross parity writes

$$GP = Re^Z e^Y.$$

Non callable convertible bond. V solves the LCP

$$\begin{pmatrix} \mathcal{L} V = 0 \\ V - \max(K_p, GP) \geq 0 \end{pmatrix} \vee \begin{pmatrix} \mathcal{L} V \leq 0 \\ V - \max(K_p, GP) = 0 \end{pmatrix} \tag{8}$$

Hard callable convertible bond. There are two cases:

- $Z + Y \geq \ln(K_c/R)$. The convertible value is simply $V = GP$ since the holder would choose to convert immediately.
- $Z + Y < \ln(K_c/R)$. In this case, the LCP writes

$$\begin{pmatrix} \mathcal{L} V = 0 \\ V - \max(K_p, GP) \geq 0 \\ V - K_c \leq 0 \end{pmatrix} \vee \begin{pmatrix} \mathcal{L} V \leq 0 \\ V - \max(K_p, GP) = 0 \\ V - K_c < 0 \end{pmatrix} \vee \begin{pmatrix} \mathcal{L} V \geq 0 \\ V - \max(K_p, GP) \geq 0 \\ V - K_c = 0 \end{pmatrix} \tag{9}$$

Soft callable convertible bond. There are two cases:

- $Z \geq \ln(T)$
  - $\diamond$ $Z + Y \geq \ln(K_c/R)$. The convertible value is simply $V = GP$ since the holder would choose to convert immediately.
  - $\diamond$ $Z + Y < \ln(K_c/R)$. The LCP writes then

$$\begin{pmatrix} \mathcal{L} V = 0 \\ V - \max(K_p, GP) \geq 0 \\ V - K_c \leq 0 \end{pmatrix} \vee \begin{pmatrix} \mathcal{L} V \leq 0 \\ V - \max(K_p, GP) = 0 \\ V - K_c < 0 \end{pmatrix} \vee \begin{pmatrix} \mathcal{L} V \geq 0 \\ V - \max(K_p, GP) \geq 0 \\ V - K_c = 0 \end{pmatrix} \tag{10}$$

- $Z < \ln(T)$

$$\begin{pmatrix} \mathcal{L} V = 0 \\ V - \max(K_p, GP) \geq 0 \end{pmatrix} \vee \begin{pmatrix} \mathcal{L} V \leq 0 \\ V - \max(K_p, GP) = 0 \end{pmatrix} \tag{11}$$

In all cases, the initial condition writes

$$V(Z, Y, 0) = \max(F, GP). \tag{12}$$

3.2 Computation domain

We are interested now in Ω, the computation domain where we will look for the solution V. In this section, $-Y_{min}$, $-Z_{min}$ and C_{max} are large numbers to numerically approximate $+\infty$.

Soft callable convertible bond. In this case, we have a strictly positive call price K_c and a strictly positive trigger T. The computation domain becomes then:

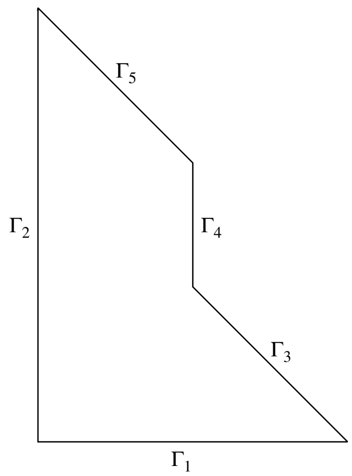

Figure 1: Ω **in the soft call case**

where Γ_1, Γ_2, Γ_3, Γ_4 and Γ_5 are respectively defined by $\{Y = Y_{min}\}$, $\{Z = Z_{min}\}$, $\{Z + Y = \ln(K/R)\}$, $\{Z = \ln(T)\}$ and $\{Z + Y = C_{max}\}$.

We know that on Γ_3 as on Γ_4, $V(Z, Y, t^*) = GP$. On Γ_1 and Γ_2, we have no Dirichlet condition, then we use Neuman conditions and we have on Γ_1

$$\frac{\partial V}{\partial n_{|\Gamma_1}} = -\frac{\partial V}{\partial Y}(Z, Y_{min}, t^*) = 0,$$

and on Γ_2,

$$\frac{\partial V}{\partial n_{|\Gamma_2}} = -\frac{\partial V}{\partial Z}(Z_{min}, Y, t^*) = 0.$$

Lastly, on Γ_5, $V_{|\Gamma_5} = GP$.

Hard Callable convertible bond. In this case, we have a strictly positive call price K_c but no trigger, and the domain becomes:

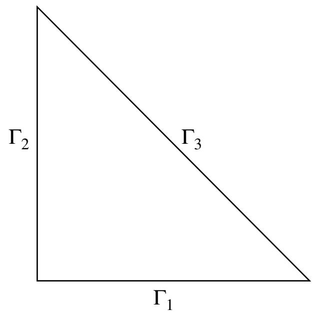

Figure 2: Ω **in a hard call case**

where Γ_1, Γ_2 and Γ_3 are respectively defined by $\{Y = Y_{min}\}$, $\{Z = Z_{min}\}$ and $\{Z + Y = \ln(K/R)\}$.

On Γ_1 we have

$$\frac{\partial V}{\partial n_{|\Gamma_1}} = -\frac{\partial V}{\partial Y}(Z, Y_{min}, t^*) = 0,$$

on Γ_2 we have

$$\frac{\partial V}{\partial n_{|\Gamma_2}} = -\frac{\partial V}{\partial Z}(Z_{min}, Y, t^*) = 0,$$

and on Γ_3, $V_{|\Gamma_3} = GP$.

Non callable convertible bond. In this case, we have no call provision and the domain becomes:

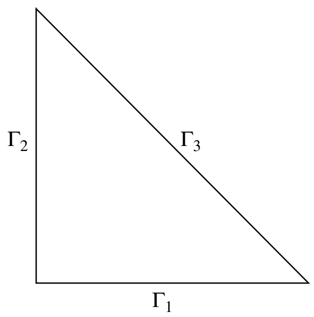

Figure 3: Ω **in case without call**

where Γ_1, Γ_2 and Γ_3 are respectively defined by $\{Y = Y_{min}\}$, $\{Z = Z_{min}\}$ and $\{Z + Y = C_{max}\}$.
On Γ_1 we have

$$\frac{\partial V}{\partial n_{|\Gamma_1}} = -\frac{\partial V}{\partial Y}(Z, Y_{min}, t^*) = 0,$$

on Γ_2 we have

$$\frac{\partial V}{\partial n_{|\Gamma_2}} = -\frac{\partial V}{\partial Z}(Z_{min}, Y, t^*) = 0,$$

and on Γ_3, we have $V_{|\Gamma_3} = GP$.

Remark 1: Reduction in the dimension.
For a cross-currency convertible bond **without soft calls and without cash dividends**, *we can reduce the problem to a pricing of a convertible bond with one factor. Indeed, after a change of variable,* $\mathcal{L}_0 V = 0$ *becomes, noting* $\xi = SX$,

$$\frac{\partial V}{\partial t} + \frac{1}{2}(\sigma^2 + 2\rho\sigma\sigma_X + {\sigma_X}^2)\xi^2\frac{\partial^2 V}{\partial \xi^2} + r\xi\frac{\partial V}{\partial \xi} - rV = 0\,.$$

We recognize the PDE followed by a one factor convertible bond with a volatility equal to $\sqrt{\sigma^2 + 2\rho\sigma\sigma_X + {\sigma_X}^2}$. *The constraints are also the same as the one factor case. Therefore, we can price the cross-currency convertible bond in this case with standard one factor convertible bond pricer with share value SX and volatility* $\sqrt{\sigma^2 + 2\rho\sigma\sigma_X + {\sigma_X}^2}$.

Having defined the domain and the boundary conditions, we now turn to the numerical analysis of our problem.

4 Numerical analysis

We now describe how we solve numerically our problem. There are several methods to solve such a problem, for example the SOR method (see Golub and Van Loan (1996) or Paul Wilmott, Jeff Dewynne and Sam Howison (1993)). We use here a method which needs the solution of the equation

$$\mathcal{L}V = 0. \tag{13}$$

Our framework is the finite element method.

Multiplying Equation (13) by a test function φ and integrating on Ω, we obtain, after integration by part and taking into account the boundary conditions, the variational formulation:

$$\begin{aligned}
&-\int_\Omega \frac{\partial V}{\partial t^*}\varphi dZdY - \frac{1}{2}\sigma_X^2\int_\Omega \frac{\partial V}{\partial Y}\frac{\partial \varphi}{\partial Y}dZdY - \rho\sigma\sigma_X\int_\Omega \frac{\partial V}{\partial Z}\frac{\partial \varphi}{\partial Y}dZdY \\
&-\frac{1}{2}\sigma^2\int_\Omega \frac{\partial V}{\partial Z}\frac{\partial \varphi}{\partial Z}dZdY + \alpha(T_f - t^*)\int_\Omega \frac{\partial V}{\partial Y}\varphi dZdY \\
&+\beta(T_f - t^*)\int_\Omega \frac{\partial V}{\partial Z}\varphi dZdY - r(T_f - t^*)\int_\Omega V\varphi\, dZdY = 0
\end{aligned} \tag{14}$$

Space discretization.
We first triangulate our computation domain Ω and build a uniform mesh which we refine, with the needed refinement degree, in an area of interest. Let N be the number of degrees of freedom which are the nodes with no Dirichlet condition where the price must be computed. We take $\varphi_j, j = \{1, \cdots, N\}$ as a basis of our discrete variational space with

$$\varphi_j(Z, Y) = \begin{cases} 1 \text{ if } (Z, Y) = j^{\text{th}} \text{ degree of freedom} \\ 0 \text{ if } (Z, Y) = \text{ an other degree of freedom} \\ \text{polynomial of degree one on every triangle} \end{cases}$$

Let $V(Z, Y, t^*) = \sum_{i=1}^N V_i(t^*)\varphi_i(Z, Y)$. For all $j \in \{1, \cdots, N\}$, Equation (14) give us, taking $\varphi = \varphi_j$,

$$\begin{aligned} &-\sum_{i=1}^N \frac{\partial V_i(t^*)}{\partial t^*} \int_\Omega \varphi_i\varphi_j dZdY - \frac{1}{2}\sigma_X^2 \sum_{i=1}^N V_i(t^*) \int_\Omega \frac{\partial \varphi_i}{\partial Y}\frac{\partial \varphi_j}{\partial Y} dZdY \\ &- \rho\sigma\sigma_X \sum_{i=1}^N V_i(t^*) \int_\Omega \frac{\partial \varphi_i}{\partial Z}\frac{\partial \varphi_j}{\partial Y} dZdY - \frac{1}{2}\sigma^2 \sum_{i=1}^N V_i(t^*) \int_\Omega \frac{\partial \varphi_i}{\partial Z}\frac{\partial \varphi_j}{\partial Z} dZdY \\ &+ \alpha(T_f - t^*) \sum_{i=1}^N V_i(t^*) \int_\Omega \frac{\partial \varphi_i}{\partial Y}\varphi_j dZdY \\ &+ \beta(T_f - t^*) \sum_{i=1}^N V_i(t^*) \int_\Omega \frac{\partial \varphi_i}{\partial Z}\varphi_j dZdY \\ &- r(T_f - t^*) \sum_{i=1}^N V_i(t^*) \int_\Omega \varphi_i\varphi_j \, dZdY = 0 \end{aligned} \tag{15}$$

In a matrix form, it writes

$$M\frac{\partial V}{\partial t^*}(t^*) + KV(t^*) = 0 \tag{16}$$

with $V(t^*)$ a vector with size N, and M and K matrix with dimension $N \times N$.

Let $i, j \in \{1, \ldots, N\}$, we have

$$\begin{aligned} M_{ij} &= \int_\Omega \varphi_i\varphi_j dZdY \text{ and} \\ K &= \frac{1}{2}\sigma_X{}^2 D^{YY} + \frac{1}{2}\sigma^2 D^{ZZ} + \rho\sigma\sigma_X D^{YZ} - \alpha(T_f - t^*)D^{\bullet Y} \\ &\quad - \beta(T_f - t^*)D^{\bullet Z} + r(T_f - t^*)M \end{aligned}$$

where
$(D^{YY})_{ij} = \int_\Omega \frac{\partial\varphi_i}{\partial Y}\frac{\partial\varphi_j}{\partial Y} \, dZdY$,
$(D^{ZZ})_{ij} = \int_\Omega \frac{\partial\varphi_i}{\partial Z}\frac{\partial\varphi_j}{\partial Z} \, dZdY$,
$(D^{YZ})_{ij} = \int_\Omega \frac{\partial\varphi_i}{\partial Y}\frac{\partial\varphi_j}{\partial Z} \, dZdY$,
$(D^{\bullet Y})_{ij} = \int_\Omega \varphi_i \frac{\partial\varphi_j}{\partial Y} \, dZdY$ and
$(D^{\bullet Z})_{ij} = \int_\Omega \varphi_i \frac{\partial\varphi_j}{\partial Z} \, dZdY$.

Time discretization : θ-scheme.
We discretize time in q intervals so that $t_0^* = 0$, $t_q^* = T_f$ and let V^m be the solution vector at time t_m^* $(m \in \{0, \cdots, q\})$. Let $\theta \in [0, 1]$ and $\Delta t^* = t_{m+1}^* - t_m^*$.
Matrix Equation (16) writes then

$$M\left(\frac{V^{m+1} - V^m}{\Delta t^*}\right) + K\left(\theta V^{m+1} + (1-\theta)V^m\right) = 0,$$

where: $\begin{cases} \theta = 1 : \text{implicit sheme} \\ \theta = 0 : \text{explicit sheme} \\ \theta = \frac{1}{2} : \text{Crank-Nicholson sheme} \end{cases}$

or also

$$\left(\frac{M}{\Delta t^*} + \theta K\right)V^{m+1} = \left(\frac{M}{\Delta t^*} - (1-\theta)K\right)V^m. \tag{17}$$

Computational methods.
To start, we build the space mesh and distinguish the degrees of freedom. We next build the time mesh and assemble the matrix M and K by using the fact that for a polynomial with degree 2 and for any triangle τ, taking three points of Gauss (ξ_1, ξ_2 and ξ_3) with there respective weights (λ_1, λ_2 and λ_3), we have:

$$\int_\tau P(Z, Y) \, dZdY = \sum_{i=1}^3 \lambda_i P(\xi_i) \,.$$

We have then an exact formulae to compute our integrals what avoids numerical error. Lastly, to solve our linear system, we use GMRES algorithm (see G.Golub and C. Van Loan 1996).

5 Greeks computation

We have now the price but in finance, sensitivities are at least as important as the price. We want then to compute some of these sensitivities (or greeks) which are

$$\Delta_S = \frac{\partial V}{\partial S}, \Delta_X = \frac{\partial V}{\partial X}, \Gamma_S = \frac{\partial^2 V}{\partial S^2}, \Gamma_X = \frac{\partial^2 V}{\partial X^2} \text{ and } \Gamma_{SX} = \frac{\partial^2 V}{\partial S \partial X}.$$

We first remark that for our meshes, the majority of nodes have six neighbours, the others have four or five of them. For a node not having six neighbours, we take of them one or two of more to have always six neighbours. Then, let 0 be the index of the node where we want to compute the greeks and $\{1, 2, 3, 4, 5 \text{ and } 6\}$ the index of its neighbours. For all index i of a neighbour node, we have the Taylor formulae

$$V_i = V_0 + \Delta_S(S_i - S_0) + \Delta_X(X_i - X_0) + \frac{1}{2}\Gamma_S(S_i - S_0)^2 + \frac{1}{2}\Gamma_X(X_i - X_0)^2 + \Gamma_{SX}(X_i - X_0)(S_i - S_0)$$

and in matrix form, we have

$$Ad = v \tag{18}$$

with:

$$v = \begin{bmatrix} V_1 - V_0 \\ V_2 - V_0 \\ V_3 - V_0 \\ V_4 - V_0 \\ V_5 - V_0 \\ V_6 - V_0 \end{bmatrix}, \quad d = \begin{bmatrix} \Delta_S \\ \Delta_X \\ \Gamma_S \\ \Gamma_X \\ \Gamma_{SX} \end{bmatrix}$$

and

$$A = \begin{bmatrix} S_1 - S0 & X_1 - X_0 & \frac{1}{2}(S_1 - S_0)^2 & \frac{1}{2}(X_1 - X_0)^2 & (S_1 - S_0)(X_1 - X_0) \\ S_2 - S0 & X_2 - X_0 & \frac{1}{2}(S_2 - S_0)^2 & \frac{1}{2}(X_2 - X_0)^2 & (S_2 - S_0)(X_2 - X_0) \\ S_3 - S0 & X_3 - X_0 & \frac{1}{2}(S_3 - S_0)^2 & \frac{1}{2}(X_3 - X_0)^2 & (S_3 - S_0)(X_3 - X_0) \\ S_4 - S0 & X_4 - X_0 & \frac{1}{2}(S_4 - S_0)^2 & \frac{1}{2}(X_4 - X_0)^2 & (S_4 - S_0)(X_4 - X_0) \\ S_5 - S0 & X_5 - X_0 & \frac{1}{2}(S_5 - S_0)^2 & \frac{1}{2}(X_5 - X_0)^2 & (S_5 - S_0)(X_5 - X_0) \\ S_6 - S0 & X_6 - X_0 & \frac{1}{2}(S_6 - S_0)^2 & \frac{1}{2}(X_6 - X_0)^2 & (S_6 - S_0)(X_6 - X_0) \end{bmatrix}_{6\times 5}$$

Then, we have six equations and only five unknowns, what encourages us to use the least-squares method to find d (see Golub and Van Loan (1996)).

6 Numerical results

In this section, we give some numerical results. We note S_0 the initial value of the underlying and X_0 the initial exchange rate.

Convertible bond with hard Call provision.
Let the convertible bond defined in Table 1.

TABLE 1: DATA FOR A HARD CALLABLE CONVERTIBLE BOND

T_f	5 years
K_c	113.7
R	1
F	105
r	0.05
r_f	0.00
σ	0.25
σ_X	0.10
ρ	0.9
S_0	100
X_0	1

For such a convertible bond, the mesh is

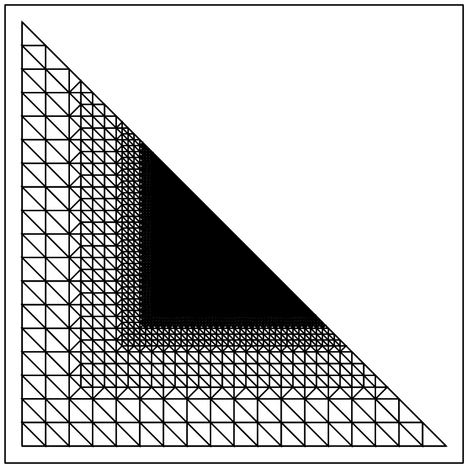

FIGURE 4: Mesh in hard call case

and the price surface is

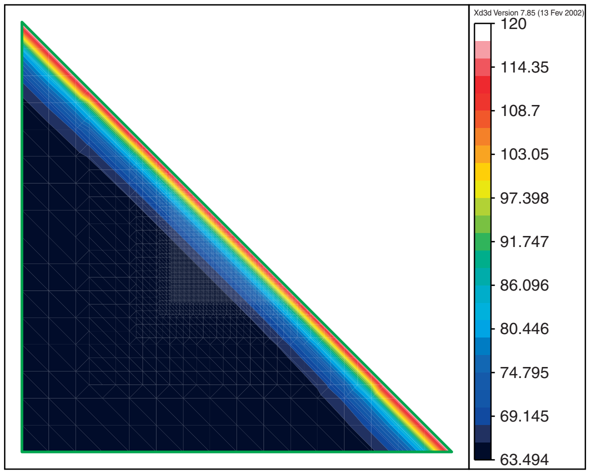

FIGURE 5: Price surface in hard call case.

We can note that the price depends on the variable SX: This confirm our Remark 1.

Convertible bond with soft Call provision.
We study the case of the convertible bond with the features described in Table 2.

TABLE 2: DATA FOR A SOFT CALLABLE CONVERTIBLE BOND

T_f	1 year
K_c	113.7
T	136.6
R	1
F	150
r	0.05
r_f	0.02
σ	0.25
σ_X	0.10
ρ	−0.9
S_0	132
X_0	1

In this case, the mesh is

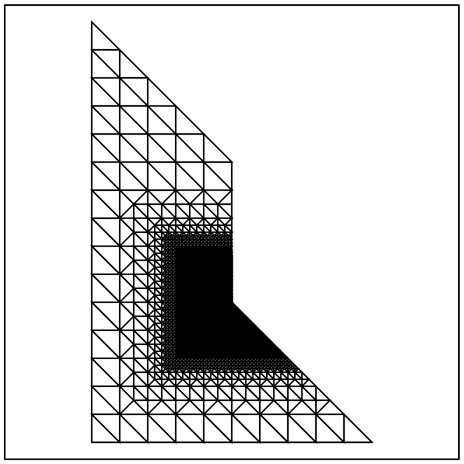

FIGURE 6: Mesh in soft call case

and the numerical results (with a constant number of timesteps equal to 100) are summerized in Table 3.

TABLE 3: PRICE AND GREEKS CONVERGENCE

Number of nodes	Price	Δ_S	Δ_X	Γ_S	Γ_X	Γ_{SX}
2357	135.5032	0.1922	120.7150	0.0202	0.3225	3.4612
3637	135.5026	0.1922	120.7074	0.0202	0.3273	3.4633
6641	135.5024	0.1922	120.7026	0.0203	0.3586	3.4643
8946	135.5023	0.1922	120.6992	0.0203	0.3518	3.4648
12272	135.5022	0.1922	120.6996	0.0203	0.3663	3.4649
15718	135.5021	0.1922	120.7006	0.0203	0.3684	3.4649

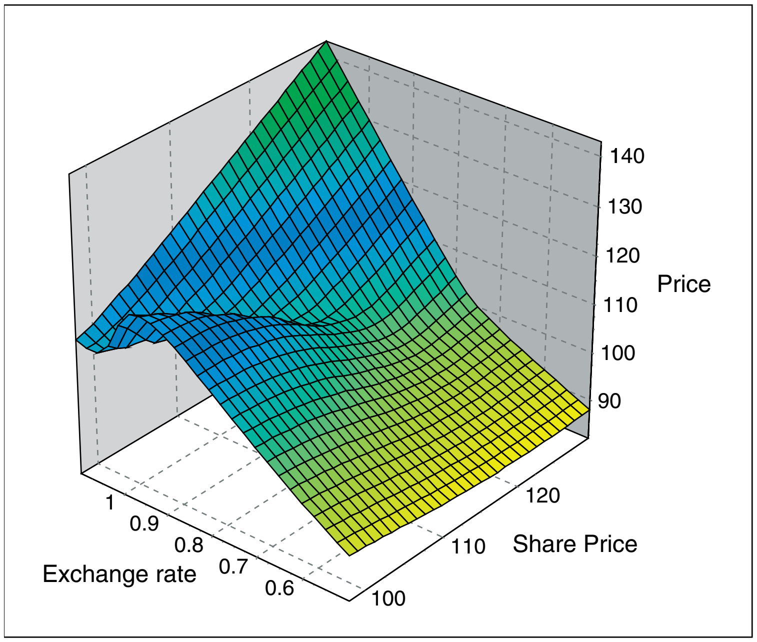

FIGURE 7: Price in a soft call case

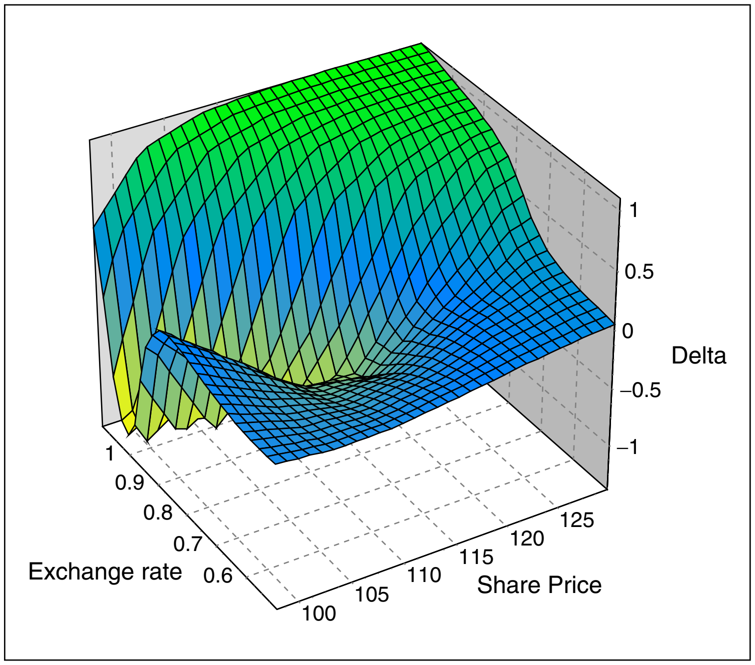

FIGURE 8: Delta in a soft call case

Table 3 demonstrates the convergence of the price and the greeks. Assuming that the currency of the convertible bond is the dollar, we note that the algorithm appear to give prices correct to \$.01 with coarse grids and greeks correct to \$.01 with moderate grids.

Our finite element solver yields the price and the greeks surfaces of the convertible bond by a unique function call. We obtain then surfaces like in Figures 7 and 8.

Conclusion

Pricing cross-currency convertible bonds cannot be reduced to a one-dimensional PDE when the soft call has a trigger in the foreign currency or when there is a cash dividend. Our two-dimensional finite element solver yields the value and the greeks of the convertible bond for all share prices and all foreign exchange rates by a unique function call whereas a two dimensional tree (or lattice) solver needs a function call for every pair (S, X).

Moreover, in cases of softly callable cross-currency convertible bonds (cases where it is not possible to reduce the problem to one dimension), trees cannot properly adress the constraints because the boundary is kinked (see figure 1). Indeed, the forced conversion boundary is complicated by the possibility of the foreign currency trading so low as to render early redemption non optimal, even for the share trading above trigger.

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